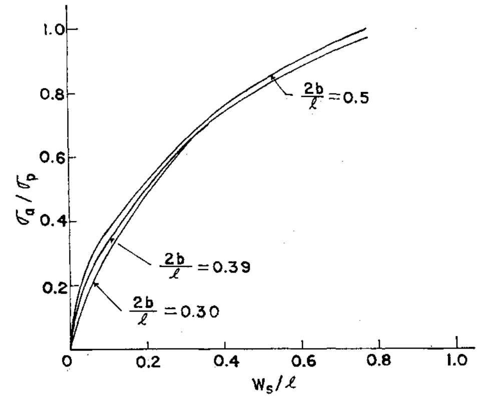

Fig. 4 Maximum admissible applied stress vs strap width.

sented in Fig. 4. Determination of more realistic applied stress distribution for fuselage panels may be found in the analysis presented in Ref. 4.

References

[1] Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1946.

[2] Wang, J. T.-S. and Hsu, T. M., "Analysis of a Panel Partially Cracked Along Midline Between Two Edges," Rep. SMN 287, April 1970, Lockheed-Georgia Co., Marietta, Ga.

[3] Crandall, S. H., *Engineering Analysis*, McGraw-Hill, New York, 1956.

[4] Wang, J. T.-S., "Orthogonally Stiffened Cylindrical Shells Subjected to Internal Pressure," *AIAA Journal*, Vol. 8, No. 3, March 1970, pp. 455–461.

Higher Vibration Modes by Matrix Iteration

L. W. Rehfield*
Georgia Institute of Technology, Atlanta, Ga.

Introduction

THE method of matrix iteration remains a useful approach to determining normal modes of vibration for elastic structures. It is straightforward to use, with or without the aid of a digital computer, and it converges rapidly if the natural frequencies are well separated. It possesses the disadvantage that numerical errors in lower vibration modes are propagated into the calculations for higher modes. This difficulty can be overcome, however, by employing a hybrid method which alternately searches for zeros in the characteristic determinant in the neighborhood of frequencies found by iteration.

The usual method of finding higher modes by matrix iteration is sweeping, which is described in textbooks on structural dynamics.[1–3] In the sweeping technique a matrix must be generated which renders any trial matrix for, say, the kth vibration mode orthogonal to the first $k-1$ modes. Thus, orthogonality of modes is assured (to within the numerical accuracy implied) and the dynamic equations are solved by iteration.

Another method for finding higher modes can be devised which satisfies both the orthogonality relations and the dynamic equations simultaneously by iteration. It stems from the result presented without proof on pages 168–169 of Ref. 1; it was communicated to these authors by M. J. Turner of the Boeing Airplane Company. This approach is the subject of this Note, and it will be referred to as "Turner's method."

Received February 17, 1972.

* Associate Professor of Aerospace Engineering. Member AIAA.

Derivation of Turner's Method

Consider a dynamic system characterized by n-degrees of freedom. Assume for illustrative purposes that the first vibration mode $\boldsymbol{\varphi}^{(1)}{}_{n\times 1}$ and its natural frequency ω_1 have been found by some means and that it is desired to find the second mode $\boldsymbol{\varphi}^{(2)}$ and its corresponding frequency ω_2 by matrix iteration. The second mode must satisfy the orthogonality relation

$$(\boldsymbol{\varphi}^{(1)T}{}_{1\times n})(M_{n\times n})\boldsymbol{\varphi}^{(2)}{}_{n\times 1} = 0 \tag{1}$$

and the dynamic equation

$$(\mathbf{D}_{n\times n})\boldsymbol{\varphi}^{(2)}{}_{n\times 1} = [1/(\omega_2)^2]\boldsymbol{\varphi}^{(2)}{}_{n\times 1} \tag{2}$$

$\mathbf{M}$ is the system's mass matrix and $\mathbf{D} = \mathbf{CM}$. $\mathbf{C}$ is the flexibility matrix for the structure.

A modified iteration problem can be defined of the following form:

$$[\mathbf{D} - (\mathbf{B}_{n\times 1})(\boldsymbol{\varphi}^{(1)})^T\mathbf{M}]\mathbf{A}_{n\times 1} = (1/\omega^2)\mathbf{A}_{n\times 1} \tag{3}$$

$\mathbf{A}$ is a matrix of modal amplitudes and $\mathbf{B}$ is a matrix whose form is as yet unspecified. Notice that if we set $\mathbf{A} = \boldsymbol{\varphi}^{(2)}$ and $\omega = \omega_2$ Equation (3) will be satisfied for any nonzero $\mathbf{B}$ matrix. We will choose a $\mathbf{B}$-matrix that will insure that the iteration of this equation will converge to $\boldsymbol{\varphi}^{(2)}$ and ω_2.

Any trial vector $\mathbf{A}$ can be expressed as a linear combination of the n true modes of the form

$$\mathbf{A} = \sum_{k=1}^{n} a_k\boldsymbol{\varphi}^{(k)} = \underset{n\times n}{\begin{bmatrix} \boldsymbol{\varphi}^{(1)}\boldsymbol{\varphi}^{(2)}\cdots\boldsymbol{\varphi}^{(n)} \\ \vdots \quad \vdots \quad\quad \vdots \end{bmatrix}} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = (\boldsymbol{\Phi}_{n\times n})(\mathbf{a}_{n\times 1}) \tag{4}$$

$\boldsymbol{\varphi}$ is the modal matrix composed of columns of vibration modes and $\mathbf{a}$ is the matrix of modal amplitudes. $\boldsymbol{\varphi}$ satisfies the equation

$$\mathbf{D\Phi} = \boldsymbol{\Phi}\begin{bmatrix} 1/(\omega_1)^2 & 0 & \mathbf{0} \\ 0 & 1/(\omega_2)^2 & \\ & & \ddots \\ & \mathbf{0} & & 1/(\omega_n)^2 \end{bmatrix} \tag{5}$$

Also

$$\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \begin{bmatrix} M_1 & 0 & \mathbf{0} \\ 0 & M_2 & \\ & & \ddots \\ & \mathbf{0} & & M_n \end{bmatrix} \tag{6}$$

where the M_k are generalized masses defined as

$$M_k = (\boldsymbol{\varphi}^{(k)})^T\mathbf{M}\boldsymbol{\varphi}^{(k)} \tag{7}$$

If Eq. (4) is substituted into the left-hand side of Eq. (3), we obtain

$$\mathbf{D\Phi}a - \mathbf{B}(\boldsymbol{\varphi}^{(1)})^T\mathbf{M\Phi a} \tag{8}$$

$$= \boldsymbol{\Phi}\begin{bmatrix} 1/(\omega_1)^2 & & & \\ & 1/(\omega_2)^2 & & \\ & & \ddots & \\ & & & 1/(\omega_n)^2 \end{bmatrix}\mathbf{a} - \mathbf{B}[\underbrace{M_1 0 0 \ldots 0}_{n\text{-terms}}]\mathbf{a}$$

$$= \left(\frac{1}{(\omega_1)^2}\,\boldsymbol{\varphi}^{(1)} - M_1\mathbf{B}\right)a_1 + \sum_{k=2}^{n}\frac{1}{(\omega_k)^2}\,\boldsymbol{\varphi}^{(k)}a_k$$

If we wish our iteration procedure to converge to the second mode, it is clear that we must eliminate any contribution due to the first mode. This is accomplished by choosing the matrix **B** so as to force the coefficient of a_1 to zero. Thus, we set

$$\mathbf{B} = [1/M_1(\omega_1)^2]\boldsymbol{\varphi}^{(1)} \tag{9}$$

and the $\boldsymbol{\varphi}^{(1)}$-contribution is removed from the iteration process. The modified equation to be iterated is

$$[\mathbf{D} - [1/M_1(\omega_1)^2]\boldsymbol{\varphi}^{(1)}(\boldsymbol{\varphi}^{(1)})^T\mathbf{M}]\mathbf{A} = (1/\omega^2)\mathbf{A} \tag{10}$$

The result of solving this equation will be second mode and frequency.

The extension of Turner's method to higher modes and frequencies is straightforward. If we have determined k modes and frequencies and wish to find the $(k+1)$-st mode, we simply iterate the following equation:

$$\left\{\left[D - \left(\sum_{j=1}^{k} \frac{1}{M_j(\omega_j)^2} \boldsymbol{\varphi}^{(j)}(\boldsymbol{\varphi}^{(j)})^T\mathbf{M}\right]\right\}\mathbf{A} = 1/\omega^2\,\mathbf{A} \tag{11}$$

Conclusions

Turner's method of matrix iteration for higher modes and frequencies has been developed. It is straightforward to apply and requires no matrix inversions for its use. It is particularly adaptable to use with digital computers and requires only the subtraction of a matrix from the dynamic matrix **D** after each successive mode and frequency are found. This method has been used with great success at Georgia Tech.

References

[1] Bisplinghoff, R. L., Ashley, H., Halfman, R. L., *Aeroelasticity*, Addison-Wesley, Reading, Mass., 1955, pp. 164–172.

[2] Hurty, W. C. and Rubinstein, M. F., *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, N.J., 1964, pp. 123–130.

[3] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, London, 1967, pp. 91–95.

Angle of Attack Increase of an Airfoil in Decelerating Flow

T. STRAND*
Air Vehicle Corporation, San Diego, Calif.

It has been found experimentally that the maximum lift coefficient of a transport airplane in flight is substantially higher than that measured in a wind tunnel when the flight airplane is undergoing an angle of attack increase while decelerating.[1]

This purpose of this Note is to determine, using inviscid theory, the aerodynamic characteristics of a two-dimensional airfoil whose angle of attack α is increasing at a constant rate $\dot{\alpha}$, and whose velocity U is decreasing at a constant rate $-\dot{U}$. Thus, let

$$U = U_0 + \dot{U}t \tag{1}$$

Received March 31, 1972.

Index Categories: Subsonic and Transonic Flow; Nonsteady Aerodynamics.

* President.

where U_0 is the velocity at $t = 0$. Without loss of generality it may be assumed that the airfoil is a flat plate. To simplify the expressions, it will be assumed that the axis of rotation of the airfoil is located at the midchord. The vertical distance η to an arbitrary point on the flat plate is then given by the following expression:

$$\eta = -x\alpha = -x(\alpha_0 + \dot{\alpha}t) \tag{2}$$

where x is the horizontal coordinate, and α_0 is the angle of attack at $t = 0$.

To obtain a solution, using the concept of acceleration potential,[2,3] it will be necessary to determine the vertical acceleration a_y of a fluid particle adjacent to the airfoil surface, i.e.,

$$a_y = \frac{d^2\eta}{dt^2} = \frac{\partial^2\eta}{\partial t^2} + 2U\frac{\partial^2\eta}{\partial x\,\partial t} + \frac{dU}{dt}\frac{\partial\eta}{\partial x} + U^2\frac{\partial^2\eta}{\partial x^2} = -2U\dot{\alpha} - \dot{U}(\alpha_0 + \dot{\alpha}t) \tag{3}$$

Let us now introduce a new coordinate system $\bar{x}$, $\bar{y}$, which is rotated with respect to the x, y system by the angle α. In this new coordinate system the flat plate airfoil lies along the abscissa. Because $\alpha \ll 1$ we find $a_y \simeq a_{\bar{y}}$. Defining a complex acceleration function $W(\bar{z})[=\phi + i\psi]$, where $\bar{z} = \bar{x} + i\bar{y}$, we have

$$a_{\bar{y}} = -\partial\psi/\partial\bar{x} = -2U\dot{\alpha} - \dot{U}(\alpha_0 + \dot{\alpha}t) \tag{4}$$

Integration with respect to $\bar{x}$ along $\bar{y} = 0$ on the airfoil yields

$$\psi = [2U\dot{\alpha} + \dot{U}(\alpha_0 + \dot{\alpha}t)]\bar{x} + C(t) \tag{5}$$

Here $C(t)$ is an integration constant. The flat plate airfoil of chord c is next mapped conformally onto the unit circle $\zeta = e^{i\theta}$, where $i = (-1)^{1/2}$, by

$$\bar{z} = (c/4)(\zeta + 1/\zeta) \tag{6}$$

Thus $\bar{x} = (c/2)\cos\theta$, $\bar{y} = 0$ for corresponding points on the flat plate and on the circle. Equation (5) therefore becomes

$$\psi = [2U\dot{\alpha} + \dot{U}(\alpha_0 + \dot{\alpha}t)](c/2)\cos\theta + C \tag{7}$$

The required acceleration function W, whose imaginary part will reduce to ψ [Eq. (7)] on $\zeta = e^{i\theta}$ and will die out at infinity, is given by

$$W = iA/\zeta + i2C/(\zeta + 1) \tag{8}$$

where

$$A = [2U\dot{\alpha} + \dot{U}(\alpha_0 + \dot{\alpha}t)]c/2 \tag{9}$$

The real part of Eq. (8) yields

$$\phi = A\sin\theta + C\tan(\theta/2) \tag{10}$$

From the general theory of acceleration potentials, we have the following expression for the lift L of an airfoil in unsteady flow

$$L = 2\rho\int_{-c/2}^{c/2}\phi\,dx \tag{11}$$

It is noted that the acceleration potential ϕ is proportional to the instantaneous chordwise pressure difference between the upper and lower surfaces of the airfoil. This being the case, we may, in order to obtain a picture of what is happening, associate the pressure difference with a fictitious effective airfoil meanline yielding this pressure distribution in a steady flow. Denoting the chordwise vortex distribution of the effective meanline by γ, the lift in steady flow becomes

$$L = \rho U\int_{-c/2}^{c/2}\gamma\,dx \tag{12}$$